Some complements on Kan complexes (based on Ex. sheet 2).

Lemma 32: (Exercise 2.4)

$X_\cdot$ Kan complex. The relation on X_0 given by: $x \sim y \iff f \in X_1, \begin{cases} d_1(f) = x \\ d_0(f) = y \end{cases}$ is an equivalence relation.

$$\Rightarrow \quad \pi_0(X) = X_0/\sim .$$

proof: Reflexivity $d_0 s_0(x) = d_1 s_0(x) = x$ (does not require $X_\cdot$ Kan)

Symmetry Assume $x \sim y$ witnessed by $p \in X_1$.

We have $\Lambda^2_0 = \Delta^1 \underset{\langle 0\rangle \quad \Delta^0 \quad \langle 0\rangle}{\coprod} \Delta^1$, i.e

$$\begin{tikzcd}
\Delta^0 \arrow[r, "\langle 0\rangle"] \arrow[d, "\langle 0\rangle"'] & \Delta^1 \arrow[d] \\
\Delta^1 \arrow[r] & \Lambda^2_0 \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

so $sSet(\Lambda^2_0, X) = \{(p_1, p_2) \in X_1^2 \mid d_1(p_1) = d_1(p_2)\}$

Consider $\Lambda^2_0 \longrightarrow X_\cdot$ given by $(p, s_0(x))$.

Apply the Kan lifting property $\Rightarrow \Delta^2 \xrightarrow{t} X.$

Then $d_0 t \in X_1$ witnesses $y \sim x$:

$$\begin{tikzcd}
 & y \arrow[dr, dashed] & \\
x \arrow[ur, "p"] \arrow[rr, "s_0(x)"'] & \equiv & x
\end{tikzcd}$$

.

Transitivity Same construction with Λ^2_1 : $\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed] & \equiv & z \end{tikzcd}$ $\square$

def 33: A simplicial set $X_\bullet$ is discrete if for all $f \in \Delta([m],[n])$, $f^*: X_n \to X_m$ is a bijection.

Lemma 33: (Exercises 1.6, 2.5)

a) The functors

$$\mathrm{Set} \longrightarrow \mathrm{sSet}^{\mathrm{discr}}$$

$$S \longmapsto \underline{S} := \text{constant presheaf with value } S:$$

$$[n] \mapsto S, \quad f \mapsto \mathrm{id}_S.$$

and $\mathrm{sSet}^{\mathrm{discr}} \xrightarrow{ev_{[0]}} \mathrm{Set}$ yield an equivalence of categories between discrete simplicial sets and sets.

b) discrete simplicial sets are Kan complexes.

proof: a) We have $ev_0(\underline{S}) \simeq S$ naturally in S.
It remains to show $\underline{ev_0(X_\cdot)} \simeq X_\cdot$ for $X_\cdot$ discrete.

There is actually a natural transformation $\underline{ev_0(-)} \xrightarrow{\varepsilon} id$ of functors $sSet \circlearrowright$, given on $X_\cdot$ by:

$$(\varepsilon_{X_\cdot})_n : \underline{ev_0(X)}_n = X_0 \xrightarrow{\langle 0 \cdots 0 \rangle^*} X_n .$$

(This is a map of s·sets because $[0]$ is the final obj. of Δ)

If $X_\cdot$ is discrete, then by definition $\varepsilon_{X_\cdot}$ is an isomorphism. and we are done. ($\underline{\text{Rmk}}$ $PSh(C)^{discr} \simeq$ Sets + action of $\pi_1(|NC|)$)

b) By part a) we can assume $X_\cdot = \underline{S}$ (we could also argue directly)

We must show: $\forall\, n \geq 1$, $\forall\, 0 \leq k \leq n$,

$$\begin{tikzcd} sSet(\Delta^n, \underline{S}) \simeq S \arrow[d, "\simeq"] \\ sSet(\Lambda^n_k, \underline{S}) \end{tikzcd}$$

I will explain in general how to compute $sSet(X_\cdot, \underline{S})$.

$\underline{\text{Claim 1}}$: A map $X_\cdot \xrightarrow{f} \underline{S}$ is determined by $X_0 \xrightarrow{f_0} S$.

$\underline{\text{Pf}}$: $\underset{n}{\underleftrightarrow{\langle 0 \cdots 0 \rangle}}^* f_n(\underset{X_n}{\underset{\cap}{x}}) = f_0(\langle 0 \cdots 0 \rangle^* \underset{X_0}{\underset{\cap}{x}})$; but $\langle 0 \cdots 0 \rangle^*$ is a bijection in $\underline{S}$.

$\underline{\text{Claim 2}}$: $f_0 : X_0 \to S$ factors through

$$\begin{tikzcd} X_0 \arrow[rr] \arrow[dr] & & S \\ & \pi_0(X) \arrow[ur, dashed] & \end{tikzcd}$$

Pf: Let $x, y \in X_0$ and $p \in X_1$, $d_1(p) = x$ and $d_0(p) = y$.

Then $\begin{cases} d_1 f_1(p) = f_0(d_1 p) = f_0(x) \\ d_0 f_1(p) = f_0(d_0 p) = f_0(y) \end{cases}$ but in $\underline{S}$, $d_0 = d_1 = id_S$!

$$\Rightarrow f_0(x) = f_0(y)$$

This is enough to finish the exercise ; indeed $\pi_0(\Lambda^n_k)$ has one element (because $I^n \subset \Lambda^n_k$, hence $0 \sim 1 \sim \cdots \sim n$).
$(n \geq 2)$

Claim 3: $sSet(X_\cdot, \underline{S}) \simeq Set(\pi_0(X_\cdot), S)$.

Pf: We have constructed an injective map $\hookrightarrow$.
It remains to see that any map $\pi_0(X_\cdot) \xrightarrow{g} S$ can be realized this way. There is a natural canditate:

$$x \in X_n \longmapsto g\left([\langle 0 \cdots 0\rangle^* x]\right) \in S$$

I leave it to you to check that this is well-defined and gives in inverse. □

Rmk:

$$\begin{tikzcd}
sSet^{\text{discr}} \arrow[dr, "|\cdot|"', "\sim"] & \simeq & Set \arrow[dl, "\sim"] \\
& Top^{\text{discr}} &
\end{tikzcd}$$

Lemma 34: (Exercise 2.6)

Trivial Kan fibrations are Kan fibrations.

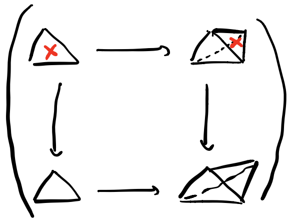
proof: Let $f : X \longrightarrow Y$ be a trivial Kan fibration, and $0 \leq k \leq n$. We are given a diagram:

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X \arrow[d] \\
\Delta^n \arrow[r] & Y
\end{tikzcd}$$

and look for a diagonal map filling it.

We have a pushout square

$$\begin{tikzcd}
\partial\Delta^{n-1} \arrow[r] \arrow[d] & \Lambda^n_k \arrow[d] \\
\Delta^{n-1} \arrow[r] & \partial\Delta^n \arrow[from=1-1, phantom, "\ulcorner", very near end]
\end{tikzcd}$$



To prove it is a pushout, can use Prop 24 on the skeletal filtration $Sk_{n-2}(\partial\Delta^n) \hookrightarrow Sk_{n-1}(\partial\Delta^n)$ is $\partial\Delta^n$

We thus have

$$\begin{tikzcd}
\partial\Delta^{n-1} \arrow[r] \arrow[d] & \Lambda^n_k \arrow[r] \arrow[d] & X \arrow[dd, "f"] \\
\Delta^{n-1} \arrow[r] \arrow[rru, dashed, "\exists"] & \partial\Delta^n \arrow[d] \arrow[ru, dashed, "\exists"] \arrow[from=1-1, phantom, "\ulcorner", very near end] & \\
& \Delta^n \arrow[r] \arrow[ruu, dashed, "\exists"'] & Y
\end{tikzcd}$$

with : $\dashrightarrow$ exists because f is a trivial Kan fibration

$\dashrightarrow$ exists because of the universal property of the pushout square.

$\dashrightarrow$ exists because f is a trivial Kan fibration.

□

II) Infinity-categories

1) Nerves of categories

Our model of ∞-categories will be a particular type of simplicial sets. We also want to be able to consider ordinary categories as ∞-categories.

$\rightsquigarrow$ We need a fully faithful functor

$$N : \mathrm{Cat} \hookrightarrow \mathrm{sSet} .$$

- Posets give rise to categories:

$$\begin{tikzcd} \mathrm{Poset} \arrow[r, hook, "\text{fully faithful}"] & \mathrm{Cat} \end{tikzcd}$$

$$P \longmapsto \mathrm{Ob} : x \in P$$

$$P(x,y) = \begin{cases} *, & x \leq y \\ \phi, & \text{otherwise} \end{cases}$$

In particular we have a fully faithful cosimplicial category:

$$\begin{tikzcd} \Delta \arrow[r, hook, "Q^{\bullet}"] & \mathrm{Cat} \end{tikzcd}$$

def 1 By the free cocompletion property, $Q^{\bullet}$ induces an adjunction:

$$\begin{tikzcd}\tau : \mathrm{sSet} \arrow[r, bend left] \arrow[r, phantom, "\perp"] & \mathrm{Cat} : N \arrow[l, bend left]\end{tikzcd}$$

Fundamental category

nerve

□

By construct°:

$$\begin{cases} \tau(\Delta^n) = [n]. \\ \tau \text{ commutes with colimits.} \end{cases}$$

and

$$N(C)_n = \mathrm{Cat}([n], C)$$

$\mathrm{sSet}(\Delta^n, N(C))$ is $\mathrm{Cat}([n], c)$

$$= \left\{ \begin{array}{l} n\text{-tuples of} \\ \text{composable morphisms} \end{array} \right\}$$

eg: $(NC)_0 = \mathrm{Ob}\, C$, $(NC)_1 = \mathrm{Mor}\, C$

$$(NC)_n = \left\{ \bullet \xrightarrow{f_1} \bullet \xrightarrow{f_2} \cdots \xrightarrow{f_n} \bullet \text{ in } C \right\}$$

Rmk We have $N([n]) \simeq \Delta^n$

$(\Leftarrow Q^{\cdot}$ fully faithful$)$.

$$N([n])_m = \mathrm{Cat}([m],[n]) \overset{\downarrow}{=} \Delta([m],[n])$$

N(C):

$$\begin{tikzcd}
\mathrm{Ob}(C) \arrow[r, gray, "s_0 = \mathrm{id}"] & \mathrm{Mor}(C) \arrow[l, bend right, "\text{target} = d_0"'] \arrow[l, bend left, "\text{source} = d_1"] \arrow[r, gray, bend left] \arrow[r, gray, bend right] & \left\{ \cdot \xrightarrow{f} \cdot \xrightarrow{g} \cdot \right\} \arrow[l, bend right, "g = d_0"'] \arrow[l, "g \circ f"', "d_1"] \arrow[l, bend left, "f = d_2"] & \cdots
\end{tikzcd}$$

This makes it clear that we can reconstruct C from $N(C)$.

Prop 2 N is fully faithful

proof: For C, D in Cat, we must show

$$\mathrm{Cat}(C,D) \xrightarrow{\sim} \mathrm{sSet}(NC, ND).$$

Injectivity: A functor is determined by its effect on objects and morphisms. Since $(NC)_0 = Ob(C)$ and $(NC)_1 = Mor(C)$, we are done.

Surjectivity Let $\alpha : NC \longrightarrow ND$.

We define a candidate F for the preimage functor using again

$$\left\{ \begin{array}{l} (NC)_0 = Ob(C) \\ (NC)_1 = Mor(C) \end{array} \right. \left| \begin{array}{l} F_{|Ob} = \alpha_0 \\ F_{|Mor} = \alpha_1 \end{array} \right.$$

To check that F is a functor, we need to see:

- F compatible with source|target:

Let $f : c \longrightarrow c'$ in C

$$\begin{cases} sF(f) \overset{def}{=} d_1\alpha(f) \overset{sSet}{=} \alpha d_1(f) \overset{def}{=} F(c) \\ tF(f) \overset{def}{=} d_0\,\alpha(f) \overset{sSet}{=} \alpha d_0(f) \overset{def}{=} F(c') \end{cases}$$

- F compatible with identities

$$F(id_c) \overset{def}{=} g\, s_0(c) \overset{sSet}{=} s_0\, g(c) \overset{def}{=} id_{F(c)}$$

- F compatible with compositions

$$F(gf) = \alpha(gf) = \alpha\big(d_1(g,f)\big)$$

$$= d_1\,\alpha(g,f) = F(g) \circ F(f)$$

$\rightsquigarrow$ F is a functor.

By construction, $N(F)$ and α coincide on 0 and 1-simplices, and it is easy to see that it forces them to be equal

(because simplices in $(NC)_i$ for $i > 2$ are uniquely determined by their 1-simplices). □

Example M monoid $\rightsquigarrow$ 1-object category BM

$N(BM)$ classying simplicial set: $N(BM)_n = M^{\times n}$

$|N(BM)|$ ——— space of M $\rightsquigarrow$ group cohomology

- We also need a precise description of fundamental categories.

Prop 3: Let $X_{\cdot} \in sSet$. The fundamental category $\tau X_{\cdot}$ admits a presentation by generators and relations:

- $\mathrm{Ob}\, \tau X_{\cdot} = X_0$
- $\mathrm{Mor}\, \tau X_{\cdot}$ is generated by X_1; for all $n \geq 0$ and $\delta_1, \dots, \delta_n \in X_1$ such that $d_1 \delta_i = d_0 \delta_{i-1}$, we have $\bar{\delta}_n \circ \cdots \circ \bar{\delta}_1 \in \mathrm{Mor}\, \tau X_{\cdot}(d_1 \delta_1, d_0 \delta_n)$.
- We have relations:
  $\forall x \in X_0$, $\overline{s_0(x)} = id_x \in \tau X(x,x)$;
  $\forall t \in X_2^{(nd)}$, we have $\overline{d_0 t} \circ \overline{d_2 t} = \overline{d_1}$.

In other words, for any other category D, we have $Cat(\tau X_{\cdot}, D)$ is in natural bijection with the set of pairs

$$\left(F_0 : X_0 \longrightarrow \mathrm{Ob}\, D, \quad F_1 : X_1 \longrightarrow \mathrm{Mor}\, D \right)$$

such that:

- $\forall \gamma \in X_1 ,\ \begin{cases} \text{source}\left(F_1(\gamma)\right) = F_0(d_1 \gamma) \\ \text{target}\left(F_1(\gamma)\right) = F_0(d_0 \gamma) \end{cases}$

$\forall x \in X_0 ,\ F_1(s_0 x) = id_{F_0(x)}.$

- $\forall t \in X_2^{(nd)},\ F_1(d_1 t) = F_1(d_0 t) \circ F_1(d_2 t)$

proof: We use the skeletal filtration together with the facts that $\begin{cases} \tau \text{ commutes with colimits.} \\ \tau(\Delta^n) = [n]. \end{cases}$

- $\tau(X_\cdot) = \operatorname{colim}_{n \geq 0} \tau(sk_n X_\cdot)$

category with one morphism.

- $\tau(sk_0 X) = \tau\Big(\coprod_{x\in X_0} \Delta^0\Big) = \coprod_{x \in X_0} [0] = \coprod_{x\in X_0} *$

- $\partial \Delta^1 = \Delta^0 \amalg \Delta^0 \Rightarrow \tau \Delta^1 = * \amalg *$

-
$$\begin{tikzcd}
\coprod_{X_1^{nd}} \partial \Delta^1 \arrow[r] \arrow[d] & \coprod_{X_1^{nd}} \Delta^1 \arrow[d] & & \coprod_{X_1^{nd}} (* \amalg *) \arrow[r] \arrow[d] & \coprod_{X_1^{nd}} [1] \arrow[d] \\
sk_0(X) \arrow[r] & sk_1(X) \arrow[ul, phantom, "\ulcorner", very near start] & \Rightarrow & \coprod_{X_0} * \arrow[r] & \tau(sk_1(X_\cdot))
\end{tikzcd}$$

$\Rightarrow \tau(sk_1(X_\cdot))$ is the free category generated by X_1^{nd}.

Since $X_1 = X_1^{nd} \amalg s_0(X_0)$, this is equivalent to generating by X_1 and imposing $\overline{s_0(x)} = id_x$.

- $\partial\Delta^2 = Sk_1(\partial\Delta^2)$

  $\Downarrow$

  $\tau(\partial\Delta^2)$ is given by:

$$\begin{tikzcd}
 & 1 \arrow[dr, "\overline{<12>}"] & \\
0 \arrow[ur, "\overline{<01>}"] \arrow[rr, bend left=10, "\overline{<12>} \circ \overline{<01>}"] \arrow[rr, bend right=10, "\overline{<02>}"'] & & 2
\end{tikzcd}$$

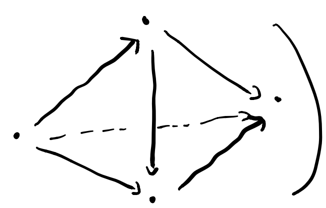
- So the map $\tau(\partial\Delta^2) \to \tau(\Delta^2) = [2]$ identifies the two morphisms between 0 and 2.

- $$\begin{tikzcd}
\coprod_{X_2^{nd}} \tau(\partial\Delta^2) \arrow[r] \arrow[d] & \coprod_{X_2^{nd}} [2] \arrow[d] \\
\tau(Sk_1 X) \arrow[r] & \tau(Sk_2 X) \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

  thus imposes exactly the relations of the statement for $t \in X_2^{nd}$.

- Any $t \in X_2^{deg}$ imposes a relation with identities which is already there.

- $$\tau(\partial\Delta^3) = \tau(Sk_2\, \partial\Delta^3)$$ ($\partial\Delta^3$ is 2-skeletal)

  $= \tau\Bigg($  $\Bigg)$

  $= [3]$ (use previous step)

$\Rightarrow \tau(\partial\Delta^3) \xrightarrow{\sim} \tau(\Delta^3)$

$\Rightarrow \forall\, X \in sSet,\ \tau(Sk_2 X) \xrightarrow{\sim} \tau(Sk_3 X.)$

skeletal filtration

$\Rightarrow \tau(\partial\Delta^4) \simeq \tau(Sk_2 \partial\Delta^4) \simeq \tau(\Delta^4)$

$\Rightarrow \forall\, X \in sSet,\ \tau(Sk_3 X) \xrightarrow{\sim} \tau(Sk_4 X.)$

$\vdots$

$\Rightarrow \forall\, X \in sSet,\ \tau(X) \xleftarrow{\sim} \tau(Sk_2 X)$ $\square$

Rmk The terminology "fundamental category" comes from topology:

- For $A \in Top$, $\tau(Sing A) \simeq \pi_{\leq 1} A$ fundamental groupoid of A.
- For $X \in Kan$, $\tau(X) \simeq \pi_{\leq 1}|X|$ ——————— $|X|$

Notation $X. \in sSet$, $x \in X_n$, $\langle \delta_0 \cdots \delta_m \rangle : [m] \to [n]$.

We write $x_{\delta_0\delta_1\cdots\delta_m} := \langle \delta_0 \cdots \delta_m \rangle^*(x) \in X_m$

In particular, $x_0, \cdots, x_n$ are the vertices of x

$x_{i,j}$ are the edges, and so on.

Thm 3 : (characterisation of nerves)

Let $X_\bullet \in sSet$. TFAE:

(1) $X_\bullet \simeq NC$ for some $C \in Cat$.

(2) The unit $X_\bullet \xrightarrow{\eta} N\tau X_\bullet$ is an isomorphism.

(3) $X_\bullet$ satisfies the unique spine extension property:

For all $n \geq 2$,

$$\begin{tikzcd}
I^n \arrow[r] \arrow[d, hook] & X_\bullet \\
\Delta^n \arrow[ur, dashed, "\exists!"'] &
\end{tikzcd}$$

Grothendieck-Segal property.

(4) $X_\bullet$ satisfies the inner horn unique extension property:

For all $n \geq 2$ and $0 < k < n$:

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & X_\bullet \\
\Delta^n \arrow[ur, dashed, "\exists !"'] &
\end{tikzcd}$$

proof: $\underline{(2) \Rightarrow (1)}$ ✓

$\underline{(1) \Rightarrow (2):}$ Assume $X \cong NC$.

By a general property of adjunctions, we have

$$\begin{tikzcd}
NC \arrow[r, "\eta N"] \arrow[dr, equal] \arrow[dr, phantom, shift left=4ex, "\circledast"] & N\tau NC \arrow[d, "N\varepsilon"] \\
 & NC
\end{tikzcd}$$

- Because N is fully faithful,

$\varepsilon : \tau N C \xrightarrow{\sim} C$ is an iso

$\Rightarrow N\varepsilon$ iso $\overset{\circledast}{\Rightarrow} \eta N$ iso

$\Rightarrow X \xrightarrow[\eta]{\sim} N\tau X$.

<u>(1) $\Rightarrow$ (3):</u>

$$\begin{array}{c}
sSet(\Delta^n, N(C)) \\
\cong \\
\{ c_0 \xrightarrow{\delta_1} c_1 \to \cdots \xrightarrow{\delta_n} c_n \text{ in } C \} \\
\cong
\end{array}$$

$$\left\{ f_1, \dots, f_n \in N(C)_1 \;\middle|\; d_0(f_i) = d_1(f_{i+1}) \right\}$$

$$\text{is} \longleftarrow I^n = \Delta^{\{0,1\}} \underset{\Delta^{\{1\}}}{\amalg} \cdots \underset{\Delta^{\{n-1\}}}{\amalg} \Delta^{\{n-1,n\}}$$

$$sSet(I^n, N(C))$$

$\underline{(3) \Rightarrow (1)}$:

We have to construct a category C and an isomorphism $X \xrightarrow{\sim} NC$.

We put $\operatorname{Ob} C = X_0$, $\operatorname{Mor} C = X_1$.

with $\operatorname{source}(f) = d_1(f)$, $\operatorname{target}(f) = d_0(f)$.

We put $\mathrm{id}_x := s_0(x)$.

Let $f, g \in X_1$, $f : x \to y$ and $g : y \to z$

we get $(f, g) : I^2 \longrightarrow X$.

By assumption there is a unique extension t to Δ^2, and we define $g \circ f$ to be $d_1(t)$:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, dashed, "g \circ f"'] & t & z \end{tikzcd}$$

The unitality then follows from the triangles

$s_0(g)$: $$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, equal, "s_0(x)"] \arrow[rr, "g"'] & & y \end{tikzcd}$$ and $s_1(g)$: $$\begin{tikzcd} & y \arrow[dr, equal, "s_0(y)"] & \\ x \arrow[ur, "g"] \arrow[rr, "g"'] & & y \end{tikzcd}$$

Associativity follows from the uniqueness of liftings along $I^3 \hookrightarrow \Delta^3$.

$\Rightarrow$ we have constructed a category C.

· We construct a morphism $X \xrightarrow{\psi} NC$.

Let $x \in X_n = sSet(\Delta^n, X)$. Then $x_{|I^n}$ determines a sequence of composable morphisms in C $\rightsquigarrow$ a simplex $\psi(x) \in (NC)_n$.

It is easy to see that $X \xrightarrow{\psi} NC$ is indeed a morphism of simplicial sets, with $\begin{cases} X_0 \xrightarrow{\sim} (NC)_0 \\ X_1 \xrightarrow{\sim} (NC)_1 \end{cases}$.

$$\begin{tikzcd} sSet(\Delta^n, X) \arrow[r, "\psi_n"] \arrow[d] & sSet(\Delta^n, NC) \arrow[d] \\ sSet(I^n, X) \arrow[r] & sSet(I^n, NC) \end{tikzcd}$$

We now contemplate the diagram:

$$\begin{tikzcd}
X_n \arrow[r] \arrow[d, "s"', "(3)"] & (N C)_n \arrow[d, "s"', "((1) \Rightarrow (3))"] \\
X_1 \times_{X_0} X_1 \times \cdots \times X_1 \arrow[r, "\sim"] & (N C)_1 \times_{(NC)_0} \cdots \times (N C)_1
\end{tikzcd}$$

and see that $X_n \xrightarrow{\sim} (N C)_n$

<u>(1) ⇒ (4):</u>

Let $n \geq 2$ and $0 < k < n$.

We compute:

$$sSet(\Lambda^n_k, N(C))$$

$$\simeq Cat(\tau(\Lambda^n_k), C)$$

$$\simeq \mathrm{Cat}\left(\tau\left(\mathrm{Sk}_2(\Lambda^n_k)\right), C\right)$$

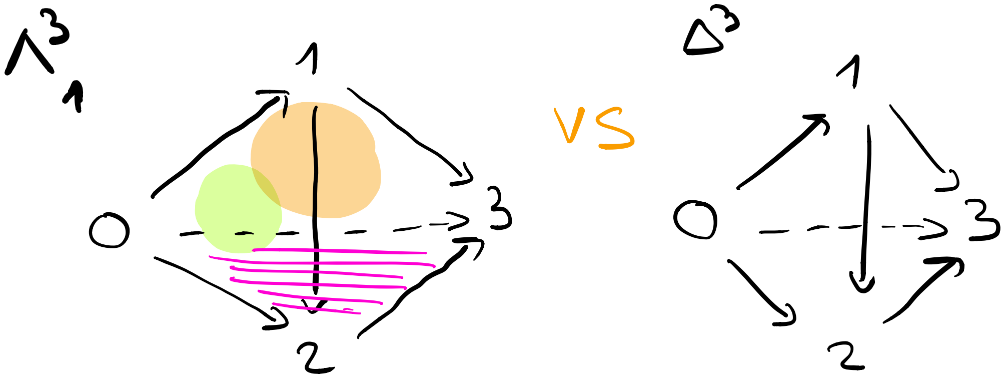
We now observe that

$$\begin{cases} \forall\, n \geq 4, \ \mathrm{Sk}_2(\Lambda^n_k) \xrightarrow{\sim} \mathrm{Sk}_2(\Delta^n) \\ \Lambda^2_1 = I^2 \ + (1) \Rightarrow (3). \end{cases}$$

So it remains to show

$$\begin{cases} \tau(\Lambda^3_1) \xrightarrow{\sim} \tau(\Delta^3) \\ \tau(\Lambda^3_2) \xrightarrow{\sim} \tau(\Delta^3) \end{cases}$$

The two cases are similar, we do Λ^3_1. First a picture:



Λ^3_1 and Δ^3 have the same objects and edges. The only additional relation in $\tau(\Delta^3)$ is

$$\overline{\langle 03\rangle} = \overline{\langle 23\rangle}\circ\overline{\langle 02\rangle}$$

but in $\tau(\Lambda^3_1)$ we have

$$\overline{\langle 03\rangle} = \overline{\langle 13\rangle}\circ\overline{\langle 01\rangle}$$

back face

$$= \left(\overline{\langle 23\rangle}\circ\overline{\langle 12\rangle}\right)\circ\overline{\langle 01\rangle}$$

$$= \overline{\langle 23\rangle}\circ\left(\overline{\langle 12\rangle}\circ\overline{\langle 01\rangle}\right)$$

left front face

$$= \overline{\langle 23\rangle}\circ\overline{\langle 02\rangle}$$

⚠ $$\begin{cases} \tau(\Lambda^3_0) \neq \tau(\Delta^3) \\ \tau(\Lambda^3_3) \neq \tau(\Delta^3) \end{cases}$$

$\Rightarrow$ $N(C)$ does not have a lifting property for <u>outer</u> horns.

<u>(4) $\Rightarrow$ (1):</u>

We follow the same strategy as for (3) $\Rightarrow$ (1). We construct a category C in exactly the same way, except that associativity is proved using

$$\Lambda^3_1 \hookrightarrow \Delta^3 \text{ rather than } I^3 \hookrightarrow \Delta^3.$$

Then we have $X \xrightarrow{\psi} NC$ defined as before with $\begin{cases} X_0 \xrightarrow{\sim} (NC)_0 \\ X_1 \xrightarrow{\sim} (NC)_1. \end{cases}$

We prove that ψ_n is a bijection

by induction on $n \geq 2$.

$$\begin{tikzcd}
X_n \arrow[r, "(5)", "\sim"'] \arrow[d] \arrow[dr, phantom, "\equiv"] & \mathrm{sSet}(\Lambda^n_1, C) \arrow[d, "✪"] \\
(N\,C)_n \arrow[r, "(1) \Rightarrow (5)", "\sim"'] & \mathrm{sSet}(\Lambda^n_1, N\,C)
\end{tikzcd}$$

But $\Lambda^n_1 = \mathrm{Sk}_{n-1}(\Lambda^n_1)$ is colimit of standard simplices of dimension $< n$, so ✪ is a bijection by induction.

□

Rmks

- We did not use the full strength of (4), only $\Lambda^n_{\text{\textcircled{1}}} \hookrightarrow \Delta^n$. For later purposes it is (4) which is relevant.

- Moreover, we saw in the course of the proof that NC has the lifting property for $\Lambda^n_k \hookrightarrow \Delta^n$, $\underline{\underline{0 \leq}}\, k \leq \underline{\underline{n}}$, as long as $n \geq 4$.

Prop 4

$$\begin{tikzcd} NC \text{ Kan complex} \arrow[d, Leftrightarrow] \arrow[r, Leftrightarrow] & \text{unique extension prop for all horns.} \arrow[dl, Leftrightarrow] \\ \text{groupoid} & \end{tikzcd}$$

proof: $\Downarrow$: Let $f : X \to Y$ be a morphism in C. We look at

$$\Lambda^2_0 \longrightarrow NC$$

given by:

$$\begin{tikzcd} & Y & \\ X \arrow[ur, "f"] \arrow[rr, equal, "\mathrm{id}_X"'] & & X \end{tikzcd}$$

By the Kan lifting property, there exists $\sigma \in (NC)_2$ with

$$\begin{tikzcd} & Y \arrow[dr, "g := d^0(\sigma)"] & \\ X \arrow[ur, "f"] \arrow[rr, equal] & \sigma & X \end{tikzcd}$$

but then $g \circ f = \mathrm{id}$.

Similarly, using $\Lambda^2_2 \hookrightarrow \Delta^2$,

$\exists\, g' : Y \to X, \quad f \circ g' = \mathrm{id}_Y$.

But then $g = g'$ and f is an isomorphism.

$$\left(g = g \circ (f \circ g') = (g \circ f) \circ g' = g' \right)$$

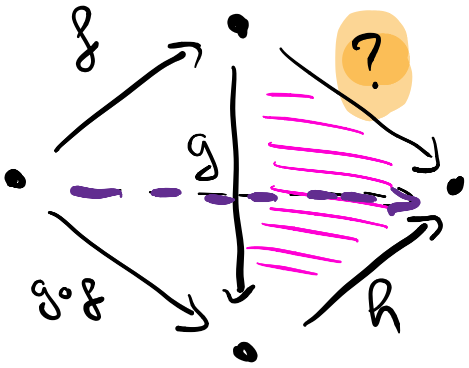
$\Rightarrow$ C groupoid.

$\underline{\Uparrow}$: By Thm 34 and the remark above, it is enough to prove the lifting property for Λ^2_0 $(, \Lambda^2_2)$, Λ^3_0 (and Λ^3_3).

Λ^2_0:

$$\begin{tikzcd} & \bullet \\ \bullet \arrow[ur, "g"] \arrow[rr, "f"'] & & \bullet \end{tikzcd} \rightsquigarrow \begin{tikzcd} & & \bullet \arrow[d, "f \circ g^{-1}"] \\ \bullet \arrow[urr, "g"] \arrow[rr, "f"'] & \equiv & \bullet \end{tikzcd}$$

Λ^3_0:



has an extension to $\Delta^3 \to NC$ iff $? = hg$.

By considering the other faces, we see that

$\dashrightarrow$: $? \circ f = h(gf)$

which is enough since f is iso. $\square$

5) ∞ - categories

def 5 An ∞ - category (or quasicategory) is a simplicial set $C_\bullet \in sSet$ satisfying the inner horn extension property: $\forall\ n \geq 2,\ \forall\ 0 < k < n,$

$$\begin{tikzcd}
\Lambda^n_k \arrow[r] \arrow[d, hook] & C_\bullet \\
\Delta^n \arrow[ur, dashed, "\exists"']
\end{tikzcd}$$

- A **functor** $F : C_\bullet \to D_\bullet$ between ∞-categories is simply a morphism of simplicial sets. This defines a (1-) category Cat^1_∞ of ∞-categories: $\mathrm{Cat}^1_\infty \xhookrightarrow{\text{full}} \mathrm{sSet}$.
- A **natural transformation** $$\begin{tikzcd} C_\bullet \arrow[r, bend left=40, "F"{name=U}] \arrow[r, bend right=40, "G"'{name=L}] & D_\bullet \arrow[Rightarrow, from=U, to=L, "\alpha"'] \end{tikzcd}$$ is a morphism $\alpha : C_\bullet \times \Delta^1 \to D_\bullet$ with $\alpha|_{C \times \{0\}} = F$ and $\alpha|_{C \times \{1\}} = G$. $\square$

Basic examples:

- By $\left\{\begin{array}{l}\text{the def. of Kan complexes}\\ \text{Prop 33 and Thm 34}\end{array}\right.$, we have fully faithful functors:

$$\begin{tikzcd} & \mathrm{Cat} \arrow[dr, hook, "N"] & & \\ \mathrm{Grp} \arrow[ur, hook] \arrow[dr, hook] & & \mathrm{Cat}^1_\infty \arrow[r, hook] & \mathrm{sSet}. \\ & \mathrm{Kan} \arrow[ur, hook] & & \end{tikzcd}$$

- We will see later that there are many examples which are not of these forms.

History

- This definition is due to Boardman-Vogt

(1973) in the context of homotopy theory (homotopy coherent algebraic structures, infinite loop spaces). They proved some basic results which we will review next.

- The idea of taking quasicategories as a model for $(\infty,1)$-categories is due to Joyal (late 90's) and he developped most of the results from the first half of this course. Then Lurie came and pushed the theory even further!

Terminology For $X_\bullet \in sSet$ (and in particular for ∞-categories), we call

- objects of $X_\bullet$ the elements of X_0
- (1-)morphisms of $X_\bullet$ —— X_1

For $g \in X_1$, we say that the source (resp. the target) of f is $d_1(g)$ (resp. $d_0(g)$) and we write $g : d_1(g) \longrightarrow d_0(g)$.

- For $x \in X_0$, we write $id_x = s_0(x)$ and call it the identity morphism of x.

Slogan: An ∞-category is like the nerve of a category, except that composition of chains of composable morphisms is only well-defined up to homotopy, and these homotopies are compatible in a precise way.

In particular, it should be possible to get a 1-category by identifying all those homotopies. This amounts to giving a simple description of the fundamental category $\tau X_\cdot$ when $X_\cdot$ is an ∞-category.

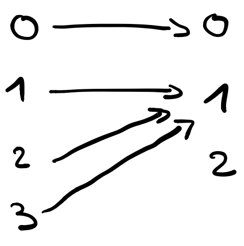
This was achieved by Boardman-Vogt.

The first tool we want to have in any "category theory" is duality.

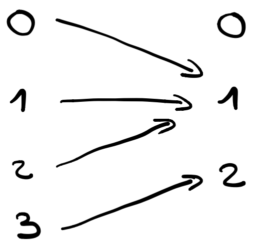
def 6 The order-reversing functor $\rho : \Delta \longrightarrow \Delta$ is defined as the identity on objects and, for $f:[m]\to[n]$,

$$\rho(f)(i) := n - f(m-i).$$

Ex:



f



$\rho(f)$

$\rho^* : sSet \longrightarrow sSet$ is the functor "precomposition by ρ". For $X_\cdot \in sSet$, the opposite simplicial set is $X_\cdot^{op} := \rho^*(X_\cdot)$. $\square$

Examples · $(\Delta^n)^{op} \simeq \Delta^n$, $(I^n)^{op} \simeq I^n$, $(\partial\Delta^n)^{op} \simeq \partial\Delta^n$.

· $(\Lambda^n_i)^{op} \simeq \Lambda^n_{n-i}$

· $X_\cdot$ ∞-category (resp Kan)

$\Longleftrightarrow$ $X_\cdot^{op}$ ∞-category (resp Kan).

- $N(C^{op}) \simeq N(C)$ for $C \in Cat$.

We now turn to the Boardman-Vogt result.

Def 7: Let $X_\bullet \in sSet$, $x, y \in X_0$.

We say that $f, g : x \to y$ are left homotopic (resp. right homotopic), written $f \sim_\ell g$ (resp. $f \sim_r g$) if $\exists\, t \in X_2$

with:

$$\begin{tikzcd} & x \arrow[dr, "g"] & \\ x \arrow[ur, "id_x"] \arrow[rr, "f"'] & t & y \end{tikzcd}$$

(resp.

$$\begin{tikzcd} & y \arrow[dr, "id_y"] & \\ x \arrow[ur, "g"] \arrow[rr, "f"'] & t & y \end{tikzcd}$$

)

$\square$

Left and right homotopy are not necc. equivalence relations; however we have

Lemma 8 Let C be an ∞-category. Then left and right homotopy coincide and is an equivalence relation.

Proof: Let $f, g, h : x \to y$ in $C(x, y)$.

We prove:

a) $f \underset{\ell}{\sim} f$.

b) $f \underset{\ell}{\sim} g$ and $g \underset{\ell}{\sim} h$ imply $f \underset{\ell}{\sim} h$.

c) $f \underset{\ell}{\sim} g$ implies $f \underset{r}{\sim} g$.

d) $f \underset{r}{\sim} g$ implies $g \underset{\ell}{\sim} f$.

<u>a)</u>: $t := f_{001}$ works:

$$\begin{tikzcd}
 & x \arrow[dr, "f"] & \\
x \arrow[ur, "\mathrm{id}_x"] \arrow[rr, "f"'] & f_{001} & y
\end{tikzcd}.$$

<u>b), c), d)</u>: They are proven in the same way:

- construct from the given 2-simplices a map

$$\Lambda^3_i \longrightarrow C, \text{ with } i = \begin{cases} 1, & \text{for b) \& c)} \\ 2, & \text{for d)} \end{cases}$$

- appeal to the inner horn extension property to get $\Delta^3 \to C$.
- restrict to the "new" 2-simplex.

It is easier in this case to display simplices so that the missing face is the back of the picture:

For b):

$$\begin{tikzcd}
x \arrow[dr, "\mathrm{id}_x"] \arrow[dd, "\mathrm{id}_x"'] \arrow[drr, bend left, "f"] \arrow[drr, phantom, "f \underset{e}{\sim} g"] & & \\
x_{000} & x \arrow[r, "g"'] \arrow[dl, "\mathrm{id}_x"] & y \\
x \arrow[urr, bend right, "h"'] \arrow[urr, phantom, "g \underset{e}{\sim} h"] & &
\end{tikzcd}$$

For c):

$$\begin{tikzcd}
x \arrow[dr, "\mathrm{id}_x"] \arrow[dd, "f"'] \arrow[drr, bend left, "g"] \arrow[drr, phantom, "g_{001}"] & & \\
f \underset{e}{\sim} g & x \arrow[r, "g"'] \arrow[dl, "g"] & y \\
x \arrow[urr, bend right, "\mathrm{id}_x"'] \arrow[urr, phantom, "g_{011}"] & &
\end{tikzcd}$$

For d):

$$\begin{tikzcd}
x \arrow[dr, "f"] \arrow[drr, bend left=40] \arrow[dd, "\mathrm{id}_x"'] & & \\
 & y \arrow[r, "\mathrm{id}_y"'] & y \\
x \arrow[ur, "f"] \arrow[urr, bend right=40, "f"'] & &
\end{tikzcd}$$

$f001$ $\qquad$ $f011$ $\qquad$ $f \underset{r}{\sim} g$

Finally:

- c), d) $\Rightarrow \underset{e}{\sim}$ is symmetric.

+ a), b) $\Rightarrow \underset{e}{\sim}$ equivalence relation

+ c), d) $\Rightarrow \underset{e}{\sim} = \underset{r}{\sim}$ $\square$

In this case we write $f \simeq g$ for $f \underset{\substack{e\\r}}{\sim} g$.

and we write $[f]$ for the homotopy class of f in $C(x,y)$.

def 9 : Let $C \in \mathrm{Cat}_\infty^1$, $f : x \to y$, $g : y \to z$ and $h : x \to z$. We say that h is $\underline{\underline{a}}$ composition of f and g if there is $t \in C_2$ with

$$\begin{tikzcd}
 & y \arrow[dr, "g"] & \\
x \arrow[ur, "f"] \arrow[rr, "h"'] \arrow[rr, phantom, shift left=3, "t"] & & z
\end{tikzcd}$$

$\square$

Prop 10 : In an ∞-category, compositions exist ; their homotopy class is well-defined and depends only on the homotopy classes of the morphisms being composed.

proof : • The existence is simply the extension property for $\Lambda^2_1 \hookrightarrow \Delta^2$.

• Let $\begin{cases} f \simeq f' : x \to y \\ g \simeq g' : x' \to y' \end{cases}$ and let $\begin{cases} h \text{ be a composition of } f \text{ and } g \\ h' \text{ ———————— } f' \text{ and } g'. \end{cases}$

We must prove that $h \simeq h'$. It is enough to treat separately the cases $f = f'$ and $g = g'$. By working in C^{op}, we reduce to $f = f'$. As before we construct a horn Λ^3_2, extends and restrict;

$$\begin{tikzcd}
x \arrow[dr, "f"] \arrow[drr, bend left, "h"] \arrow[dd, "id_x"'] & & \\
& y \arrow[r, "g"] & z \\
x \arrow[ur, "f'"'] \arrow[urr, bend right, "h'"'] & &
\end{tikzcd}$$

$\square$

Prop 11 : The resulting composition on homotopy classes of morphisms is associative and unital.

proof: same method as Prop 39: left as exercise $\square$

def 12 Let C be an ∞-category. Its homotopy category hC has:

$$\begin{cases} Ob(hC) = C_0 \\ hC(x,y) = C(x,y)/\simeq \end{cases}$$

By Prop 39-40, this is indeed a 1-category.

Prop 13: The construction of the homotopy category defines a functor $h : Cat^1_\infty \longrightarrow Cat$ which is a left adjoint to $N : Cat \longrightarrow Cat^1_\infty$.

($\Longleftrightarrow$ $\tau C \simeq hC$ naturally in C)

proof: · The functoriality follows from the fact that for $F : C \to D$, $g \simeq g'$ in $C \Rightarrow F(g) \simeq F(g')$ which is clear.

· Let C be an ∞-category. We construct a natural equivalence (in fact isomorphism) of categories $\tau C \xrightarrow[\varphi]{\sim} hC$.

Because of Prop 3, to define φ it is enough to define $\begin{cases} C_0 \longrightarrow \mathrm{Ob}\, hC \\ C_1 \longrightarrow \mathrm{Mor}\, hC \end{cases}$

satisfying some relations given by identities and C_2.

We put $C_0 = \mathrm{Ob}\, hC$

$$C_1 \longrightarrow \mathrm{Mor}\, hC \,, \quad g \longmapsto [g]$$

and the relations are satisfied by def 12.

- By construction, φ is $\begin{cases} \text{bijective on objects} \\ \text{surjective on morphisms.} \end{cases}$

It remains to show φ is faithful.

Because of the lifting property for $\Lambda^2_1 \hookrightarrow \Delta^2$, any morphism in τC can be written as $\overline{g}$ for $g \in C_1$. Now suppose that $g, g' : x \to y$ satisfy $[g] = [g']$. By definition, there is an homotopy

$$\begin{tikzcd} x \arrow[d, equal] \arrow[dr, "g'"] \arrow[dr, phantom, "\sigma"'] & \\ x \arrow[r, "g"'] & y \end{tikzcd}$$

, but this also implies $\mathrm{id}_x \circ \overline{g'} = \overline{g}$ in τC and we are done. □

Let us see some basic ways to construct new ∞ - categories.

Prop 14 : 1) Arbitrary products and coproducts of ∞ - categories (in sSet) are ∞ - categories.

2) Filtered colimits of ∞ - categories are ∞ - categories.

proof: 1)

Let $\{X_\alpha\}_{\alpha \in J}$ be a family of ∞ - categories.

. Let $0 \leq k \leq n$. We have

$$\begin{tikzcd}
sSet\left(\Delta^n, \prod_\alpha X_\alpha\right) \arrow[r] \arrow[d, phantom, "\cong" rotate=90] & sSet\left(\Lambda^n_k, \prod_\alpha X_\alpha\right) \arrow[d, phantom, "\cong" rotate=90] \\
\prod_\alpha sSet(\Delta^n, X_\alpha) \arrow[r, two heads] & \prod_\alpha sSet(\Lambda^n_k, X_\alpha)
\end{tikzcd}$$

↑ product of surjections is a surjection

$\Rightarrow \prod_\alpha X_\alpha$ is an ∞ - category.

- For the coproduct, we need to compute

$$\mathrm{sSet}\left(\Lambda^n_k, \coprod X_\alpha\right) \quad \text{and} \quad \mathrm{sSet}\left(\Delta^n, \coprod X_\alpha\right)$$

By Yoneda,

$$\mathrm{sSet}\left(\Delta^n, \coprod X_\alpha\right) = \left(\coprod_\alpha X_\alpha\right)([n])$$

$$\overset{\substack{\text{colimits}\\ \text{are objectwise}}}{=} \coprod_\alpha X_\alpha([n])$$

$$= \coprod_\alpha \mathrm{sSet}(\Delta^n, X_\alpha)$$

So it suffices to show that the natural map

$$\coprod_\alpha \mathrm{sSet}(\Lambda^n_k, X_\alpha) \longrightarrow \mathrm{sSet}\left(\Lambda^n_k, \coprod_\alpha X_\alpha\right)$$

is a bijection. For this one can use

$$\Lambda^n_k = \coprod_{\substack{\Delta^{[n]\setminus\{i,j\}} \\ i \neq k}} \Delta^{[n]\setminus i}$$

and the Yoneda trick above; the key point is that the various $\Delta^{[n]\setminus i}$ must be sent to $\underline{\underline{\text{the same}}}$ X_α because they are connected via the (n-2) - faces $\Delta^{[n]\setminus\{i,j\}}$. Or in other words, one can show that

$\pi_0\left(\coprod X_\alpha\right) \simeq \coprod_\alpha X_\alpha$ while $\pi_0(\Lambda^n_k)$ has one element (Λ^n_k is connected).

2) Let J be a filtered category and $X : J \longrightarrow sSet$ be a diagram so that each $X(\alpha)$ is an ∞-category. Once again we have $\left(\operatorname{colim}_J X\right)_n = \operatorname{colim}_J X([n])$ because colimits are computed objectwise.

We want to show that the canonical map

$$\operatorname{Colim}_{\alpha\in J} sSet\left(\Lambda^n_k, X(\alpha)\right) \longrightarrow sSet\left(\Lambda^n_k, \operatorname{Colim}_J X\right)$$

is a bijection. We will show this holds for Λ^n_k replaced by any $Y \in sSet$ with finitely many non-degenerate simplices.

Let $\mathcal{C} = \left\{ Y \in sSet \;\middle|\; \begin{array}{l} sSet(Y,-) \text{ commutes} \\ \text{with filtered colimits} \end{array} \right\}$.

As remarked above, $\Delta^n \in \mathcal{C}$ for all $n \in \mathbb{N}$.

Let's show that $\mathcal{C}$ is closed under finite colimits. Let K be a finite category and $Y : K \longrightarrow \mathcal{C}$ be a diagram.

We have

$$\begin{tikzcd}
\operatorname{Colim}_{\alpha \in J} sSet\left(\operatorname{Colim}_K Y, X(\alpha)\right) \arrow[rr] \arrow[d, dash, "\cong"', "\text{colim prop}"] & & sSet\left(\operatorname{Colim}_K Y, \operatorname{Colim}_J X\right) \arrow[d, dash, "\cong", "\text{colim prop.}"'] \\
\operatorname{Colim}_{\alpha \in J} \operatorname{Lim}_{\beta \in K} sSet\left(Y(\beta), X(\alpha)\right) \arrow[dr, dash, "\cong"', "\text{Filtered colimits commutes with finite limits in Set.}"] & & \operatorname{Lim}_{\beta \in K} sSet\left(Y(\beta), \operatorname{Colim}_J X\right) \arrow[dl, dash, "\cong", "X(\alpha) \in \mathcal{C}"'] \\
& \operatorname{Lim}_{\beta \in K} \operatorname{Colim}_{\alpha \in J} sSet\left(Y(\beta), X(\alpha)\right) &
\end{tikzcd}$$

$$\Longrightarrow \operatorname{Colim}_K Y \in \mathcal{C}.$$

- By the skeletal filtration,

$$\left\{ \begin{array}{l} \text{s.sets with } < \infty \\ \text{non-deg. simplices} \end{array} \right\} = \left\{ \begin{array}{l} \text{finite colimits of} \\ \text{standard simplices} \end{array} \right\}$$

and we are done. □